

AN APPLICATION OF THE O'NAN-SCOTT THEOREM TO THE GROUP GENERATED BY THE ROUND FUNCTIONS OF AN AES-LIKE CIPHER

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ABSTRACT. In a previous paper, we had proved that the permutation group generated by the round functions of an AES-like cipher is primitive. Here we apply the O'Nan Scott classification of primitive groups to prove that this group is the alternating group.

1. INTRODUCTION

According to Shannon [Sha49, p. 657], a cipher “is defined abstractly as a set of transformations”. Coppersmith and Grossman [CG75], and later in 1988 Kaliski, Rivest and Sherman [KRS88], called attention to the group generated by a cipher. One of the motivations for the work of Kaliski et al. is that at that time Triple DES was being suggested as an improvement to DES. This meant replacing the use of single DES transformation T_a , where a is a key, with the composition $T_a T_b T_c$, where a, b, c are three DES keys. If it was the case that the transformations of DES form a group, then Triple DES would have been of course no more than DES itself. More generally, Kaliski et al. showed that if the group generated by the transformations of a cipher is too small, then the cipher is exposed to certain cryptanalytic attacks.

It was later proved by Wernsdorf [Wer93] that the group generated by the round functions of DES (which are even permutations) is the alternating group. This implies that the group generated by the DES transformations with independent subkeys is also the alternating group. (We are not aware of any work in this context that tries to take account of the key schedule.)

Wernsdorf used ad hoc methods in [Wer02] to prove that the permutation group G generated by the round functions of AES is the alternating group. (Here, too, these functions are even permutations.) Sparr and Wernsdorf have recently given another, permutation group theoretic proof in [SW08].

The goal of this paper is to give a different proof of this fact, building upon our earlier paper [CDVS08]. There we had proved that the group G is primitive.

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In the course of doing that we answered a question of Paterson [Pat99] about the possibility of embedding a trapdoor in a cipher by having the group generated by the cipher act imprimitively.

In this paper we work under certain cryptographic assumptions (see Section 2) that are a stripped down, simplified version of those of [CDVS08]. (These are also satisfied by AES.) We first give, for the convenience of the reader, a short group-theoretic version of the main result of [CDVS08] under these assumptions. We then appeal to the O’Nan-Scott classification of primitive groups to prove that the group generated by the round functions of a cryptosystem satisfying our assumptions is the alternating group.

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2. PRELIMINARIES

In the rest of the paper, we tend to adopt the notation of [DR02].

Let $V = V(d, 2)$, the vector space of dimension d over the field $\text{GF}(2)$ with two elements, be the state (or message) space. V has $n = 2^d$ elements.

For any $v \in V$, consider the translation by v , that is the map

$$\begin{aligned}\sigma_v : V &\rightarrow V, \\ w &\mapsto w + v.\end{aligned}$$

In particular, σ_0 is the identity map on V . The set

$$T = \{ \sigma_v : v \in V \}$$

is an elementary abelian, regular subgroup of $\text{Sym}(V)$. In fact, the map

$$(2.1) \quad \begin{aligned}V &\rightarrow T \\ v &\mapsto \sigma_v\end{aligned}$$

is an isomorphism of the additive group V onto the multiplicative group T .

We consider a *key-alternating block cipher* (see Section 2.4.2 of [DR02]) which consists of a fixed number of iterations of a function of the form $\rho\sigma_k$, where $k \in V$. Such a function is called a *round function*, and the parameter k is called the *round key*. (We write maps left-to-right, so ρ operates first.) Here ρ is a fixed permutation operating on the vector space V . Therefore each round consists of an application of ρ , followed by a key addition. This covers for instance AES with *independent* subkeys. Let $G = \langle \rho\sigma_k : k \in V \rangle$ be the group of permutations of V generated by the round functions. Choosing $k = 0$ we see that $\rho \in G$, and thus $T \leq G$. It follows that $G = \langle T, \rho \rangle$.

We assume $\rho = \gamma\lambda$, where γ and λ are permutations. Here γ is a bricklayer transformation, consisting of a number of S-boxes. The message space V is written as a direct sum

$$V = V_1 \oplus \cdots \oplus V_{n_t},$$

$n_t > 1$, where each V_i has the same dimension $m > 1$ over $\text{GF}(2)$. As $n_t > 1$, this implies that $d = mn_t$ is not a prime number. For $v \in V$, we will write

$v = v_1 + \cdots + v_{n_t}$, where $v_i \in V_i$. Also, we consider the projections $\pi_i : V \rightarrow V_i$, which map $v \mapsto v_i$. We have

$$v\gamma = v_1\gamma_1 \oplus \cdots \oplus v_{n_t}\gamma_{n_t},$$

where the γ_i are S-boxes, which we allow to be different for each V_i .

λ is a linear function (usually called a linear mixing layer). The only assumption we will be making about λ is Cryptographic Assumption (3) below.

In AES the S-boxes are all equal, and consist of inversion in the field $\text{GF}(2^8)$ with 2^8 elements (see later in this paragraph), followed by an affine transformation, that is, a linear transformation, followed by a translation. When interpreting AES in our scheme, we take advantage of the well-known possibility of moving the linear part of the affine transformation to the linear mixing layer, and incorporating the translation in the key addition (see for instance [MR02]). Thus in our scheme for AES we have $m = 8$, we identify each V_i with $\text{GF}(2^8)$, and we take $x\gamma_i = x^{2^8-2}$, so that γ_i maps nonzero elements to their inverses, and zero to zero. As usual we will simply say that γ_i acts by inversion.

We will work under the following

Cryptographic Assumptions. *Consider an AES-like cryptosystem as described above, which satisfies the following conditions.*

- (1) $0\gamma = 0$ and $\gamma^2 = 1$, the identity transformation.
- (2) *There is $1 \leq r < m/2$ such that the following hold.*
 - (a) *For all $0 \neq v \in V_i$, the image of the map $V_i \rightarrow V_i$, which maps $x \mapsto (x+v)\gamma_i + x\gamma_i$, has size greater than 2^{m-r-1} , and it is not a coset of a subspace.*
 - (b) *There is no subspace of V_i , invariant under γ_i , of codimension less than or equal to $2r$.*
- (3) *There are no subspaces U, U', U'' (except $\{0\}$ and V) that are the sum of some of the V_i , and such that $U\lambda = U'$ and $U'\lambda = U''$.*

In [CDVS08] we have proved under certain abstract and general assumptions a result that specializes to the following:

Theorem 1. *Suppose a cryptosystem satisfies the Cryptographic Assumptions. Then the group G generated by its round functions acts primitively on the message space V .*

We give a short, group-theoretic proof of this in Section 3. This we do for the convenience of the reader, as we will need to refer to part of the proof in Section 6. We are grateful to the referee of another paper for this proof.

In the rest of the paper we prove the following

Theorem 2. *Suppose a cryptosystem satisfies the Cryptographic Assumptions.*

Then the group G generated by its round functions is the alternating group $\text{Alt}(V)$.

The same holds for the group generated by the cryptosystem with independent subkeys.

A word about the parity of the group G is in order here. Over $V = V(d, 2)$, non-trivial translations are clearly involutions without fixed points, and thus even permutations. Also, for $d > 2$ the group $\text{GL}(d, 2) = \text{SL}(d, 2)$ is perfect, so that in particular it has no (normal) subgroup of order 2, and it is thus contained in $\text{Alt}(V)$.

We now show that γ is also even, so that $G \leq \text{Alt}(V)$. In fact, γ is the product of n_t permutations g_i , acting as γ_i on V_i , and as the identity on V_j , $j \neq i$. This means that every 2-cycle in γ_i gives rise to 2^{d-m} 2-cycles in g_i . Now the number 2^{d-m} is even, as $d - m = n_t m - m > m$, $n_t > 1$ by assumption, and $m > 2$ by Cryptographic Assumption (1). It follows that each g_i is even, and thus so is γ . (The same argument proves that γ is even, even without assuming that it is an involution, as we do here.)

Condition (1) is clearly satisfied by AES. As we said above, we take advantage here of the possibility of assuming that γ is simply componentwise inversion.

Condition (2a) is also well-known to be satisfied, with $r = 1$ (see [Nyb94] but also [DR06]), as the image of that map has size $2^7 - 1$.

As to Condition (2b), it is also satisfied by AES with $r = 1$. For that, one could just use GAP [GAP05] to verify that the only nonzero subspaces of $\text{GF}(2^8)$ which are invariant under inversion are the subfields. However, this can also be derived from a more general result of [GGSZ06] and [Mat07], which states that the only nonzero additive subgroups of $\text{GF}(2^m)$, which contain the inverse of all of their nonzero elements, are the subfields.

Condition (3) follows from the properties of the components **MixColumns** [DR02, 3.4.3] and **ShiftRows** [DR02, 3.4.2] of the linear mixing layer (which are not altered by the fact that we have incorporated in it the linear part of the S-boxes). In fact, suppose, without loss of generality, that $U \supseteq V_1$. Then U' contains the whole first column of the state, and $U'' = V$, a contradiction. This argument is a vestigial form of the Four-Round Propagation Theorem [DR02, 9.5.1].

3. PRIMITIVITY

In this section we give a proof of Theorem 1.

Suppose for a contradiction that $G = \langle T, \rho \rangle$ is imprimitive on V , so that any block system for G is given by the cosets of some subspace U of V . This is because, as it is proved in [CDVS08], a block system for G is also a block system for the group T of translations.

Now $\rho = \gamma\lambda$, with λ linear, and $0\gamma = 0$. Thus $U\rho = U$, and $U' = U\gamma = U\lambda^{-1}$ is a subspace.

Suppose firstly that $U = V_{i_1} \oplus \cdots \oplus V_{i_l}$ is a direct sum of some of the subspaces V_i ($l < n_t$). Then, $U' = U\gamma = U$, so that $U' = U$ is λ -invariant; this contradicts Cryptographic Assumption (3).

Thus there exists i such that $U \not\supseteq V_i$, but there is $u \in U$, such that its i -th component $u_i \in V_i$ is nonzero. We claim that $U \cap V_i$ is nonzero. Take any $v \in V_i$. Then $(u + v)\gamma + v\gamma \in U'$, so that $u\gamma + (u + v)\gamma + v\gamma \in U'$. The latter element has all zero components, except possibly the i -th one, which is $u_i\gamma_i + (u_i + v)\gamma_i + v \in U' \cap V_i$. Were the latter zero for all $v \in V_i$, then the map

$V_i \rightarrow V_i$ that maps $v \mapsto (u_i + v)\gamma_i + v\gamma_i$ would be constant, thus contradicting Cryptographic Assumption (2a).

Thus there exists i such that both $U_i = U \cap V_i$ and $U'_i = (U_i)\gamma_i = U' \cap V_i$ are nonzero, proper subspaces of V_i of the same dimension, and

$$\gamma_i : V_i/U_i \rightarrow V_i/U'_i.$$

If $x \in V_i$, and $v \in U_i$, $v \neq 0$, then $x + v$ and x are in the same coset of U_i , so $(x + v)\gamma_i$ and $x\gamma_i$ are in the same coset of U'_i . Thus the set

$$\{(x + v)\gamma_i + x\gamma_i : x \in V_i\}$$

is a subset of U'_i , and by Cryptographic Assumption (2a) U_i and U'_i have size greater than 2^{m-r-1} , that is to say dimension at least $m - r$ or equivalently codimension at most r . The codimension of $U_i \cap U'_i$ is therefore at most $2r$, so $U_i \cap U'_i$ cannot be γ_i -invariant because of Cryptographic Assumption (2b). This means there exists $z \in U_i \cap U'_i$ such that $z\gamma_i \notin U_i \cap U'_i$, so $z\gamma_i \notin U_i$, as $z\gamma_i \in U'_i$. However, U'_i is the image of U_i under the bijective map γ_i , so $z = z\gamma_i^2 \notin U'_i$, as $z\gamma_i \notin U_i$. Thus $z \notin U_i \cap U'_i$, which is a contradiction.

4. O'NAN-SCOTT

In this section we prove Theorem 2. We first state the O'Nan-Scott classification of primitive groups for the case of the maximal primitive subgroups of the symmetric group. We give the result for the symmetric group of degree q^n , where q is a power of a prime number p .

Theorem 3. [Cam99, Theorem 4.8] *Suppose q is a power of the prime p .*

A maximal primitive subgroup G of $\text{Sym}(q^n)$ is one of the following:

- (1) *affine, that is, $G = \text{AGL}(d, p)$, $p^d = q^n$, for some d ;*
- (2) *primitive non-basic, that is, a wreath product $G = \text{Sym}(k) \wr \text{Sym}(r)$ in product action, $k^r = q^n$, $k \neq 2$, $r > 1$.*
- (3) *almost simple, that is, $S \leq G \leq \text{Aut}(S)$, for a nonabelian simple group S .*

Note that in our context $p = 2$.

It is convenient to use a refinement of the O'Nan-Scott theorem, due to Cai Heng Li [Li03], for the special case when G contains an abelian regular subgroup T ; in our case, this is the group of translations.

Theorem 4. [Li03, Theorem 1.1] *Let G be a primitive group of degree 2^d , with $d \geq 1$. Suppose G contains a regular abelian subgroup T .*

Then G is one of the following

- (1) *affine, that is, $G \leq \text{AGL}(d, 2)$;*
- (2)

$$G = (S_1 \times \cdots \times S_r).O.P,$$

with $2^d = m^r$ for some m and $r > 1$. Here $T = T_1 \times \cdots \times T_r$, with $T_i < S_i \cong \text{Alt}(m)$ for each i , $O \leq \text{Out}(S_1) \times \cdots \times \text{Out}(S_r)$, and P permutes transitively the S_i .

- (3) *almost simple, that is, $S \leq G \leq \text{Aut}(S)$, for a nonabelian simple group S .*

To prove the first statement of Theorem 2 we need to deal with the three possible cases of Theorem 4.

Case (1) is treated in Section 5. An important observation of Li [Li03] is in order here. If V is a vector space, with addition $+$, then the symmetric group $\text{Sym}(V)$ contains the affine group $\text{AGL}(V) = T \text{GL}(V)$, where T is the group of translations. But $\text{Sym}(V)$ also contains the conjugates of $\text{AGL}(V)$, which are still affine groups on the set V , but possibly with respect to an operation \circ different from $+$. In particular the group T of translations may be contained in one of these conjugates, where it will be an abelian regular subgroup. We have studied this situation in [CDVS06], and we will be exploiting these results in Section 5.

Case (2) will be dealt with in Section 6.

In the almost simple case (3), the intersection of a one-point stabilizer in G with S is a proper subgroup of S of index 2^d , since the nontrivial normal subgroup S of the primitive group G is transitive. We can thus appeal (as Li does) to a particular case of a result of Guralnick [Gur83], which states that the only nonabelian simple groups that have a subgroup of index of the form 2^d are either the alternating groups $S = \text{Alt}(2^d)$, with $d > 2$, or the groups $\text{PSL}(f, q)$, where q is a prime-power, and f is prime, $(q^f - 1)/(q - 1) = 2^d$. We rule out the second possibility as follows. Since $(q^f - 1)/(q - 1) = q^{f-1} + q^{f-2} + \dots + q + 1 \equiv f \pmod{2}$, we have $f = 2$ here, and $q = 2^d - 1$. Well-known elementary arguments yield that q and d are prime. However, $d = n_t m$ is not prime, as $n_t > 1$ by assumption, and as noted earlier $m > 2$ by Cryptographic Assumption (1).

Clearly $\text{Aut}(\text{Alt}(2^d)) = \text{Sym}(2^d)$ here, so G is either the alternating or the symmetric group. Since we have shown in Section 2 that $G \leq \text{Alt}(V)$, we obtain $G = \text{Alt}(V)$.

To prove the second statement of Theorem 2, we then appeal to a standard argument: if the nonabelian simple group G is generated by a subset S , then for any fixed r the set $S' = \{s_1 s_2 \dots s_r : s_i \in S\}$ of r -fold products of elements of S generates a nontrivial normal subgroup of G , and thus S' also generates G . In our context S is the set of the round functions for all possible subkeys, and r is the number of rounds, so that S' is the set of the transformations of the cryptosystem with independent subkeys.

5. THE AFFINE CASE

Suppose G is contained in an affine subgroup of $\text{Sym}(V)$. By the theory of [CDVS06], there is a structure of an associative, commutative, nilpotent ring $(V, \circ, \cdot, 0)$ on V , such that $(V, \circ, 0)$ is a vector space over the field with two elements, and ordinary addition on V is expressed as

$$x + y = x \circ y \circ xy,$$

for $x, y \in V$. Moreover, G acts as a group of affine transformations on $(V, \circ, 0)$.

As both $(V, \circ, 0)$ and $(V, +, 0)$ are elementary abelian, we have

$$0 = x + x = x \circ x \circ xx = 0 \circ x^2 = x^2$$

for all $x \in V$. It follows

$$\begin{aligned} x + y + xy &= (x \circ y \circ xy) \circ xy \circ (x \circ y \circ xy) \cdot xy \\ &= x \circ y \circ xy \circ xy \circ x^2 y \circ xy^2 \circ x^2 y^2 \\ &= x \circ y. \end{aligned}$$

Here we have used the fact that \cdot distributes over \circ .

Now $\rho \in G$ is linear with respect to \circ , that is $(x \circ y)\rho = x\rho \circ y\rho$ for all $x, y \in V$. Choose $0 \neq y \in U = \{z \in V : xz = 0 \text{ for all } x \in V\}$. (The latter set is different from $\{0\}$, as the ring $(V, \circ, \cdot, 0)$ is nilpotent.) Then

$$(5.1) \quad (x + y)\rho = (x \circ y)\rho = x\rho \circ y\rho = x\rho + y\rho + x\rho \cdot y\rho.$$

Now note that given $x \in V$, the set $xV = \{xz : z \in V\}$ is a subspace with respect to \circ , as \cdot distributes over \circ ; and also a subspace with respect to $+$, as $xz_1 + xz_2 = xz_1 \circ xz_2 \circ x^2 z_1 z_2 = xz_1 \circ xz_2$.

It follows from 5.1 that for $0 \neq y \in U$ we have

$$\{(x + y)\rho + x\rho : x \in V\} = y\rho + y\rho V.$$

The right hand side is a coset of a subspace of V with respect to $+$. Now λ (and its inverse) are linear with respect to $+$. Applying λ^{-1} we obtain that

$$\{(x + y)\gamma + x\gamma : x \in V\}$$

is also a coset of a subspace of V with respect to $+$. Choose an index i so that the component $y_i \in V_i$ of y is nonzero. Then we have that the projection on V_i of the previous set

$$\{(x + y_i)\gamma + x\gamma : x \in V_i\}$$

is a coset of a subspace of V_i with respect to $+$. This contradicts Cryptographic Assumption (2a).

6. WREATH PRODUCT IN PRODUCT ACTION

Here we deal to the case when

$$G = (S_1 \times \cdots \times S_r).O.P,$$

with $2^d = k^r$ for some k and $r > 1$. Here $T = T_1 \times \cdots \times T_r$, where $|T_i| = k$ and $T_i < S_i \cong \text{Alt}(k)$ for each i , $O \leq \text{Out}(S_1) \times \cdots \times \text{Out}(S_r)$, and P permutes transitively the S_i by conjugation. It follows that $S_1 \times \cdots \times S_r = \text{Soc}(G)$.

Note that if $k = 2$ or 4 , so that $S_i \cong \text{Alt}(2)$ or $\text{Alt}(4)$, the group T of translations is normal in G , so that $G \leq \text{AGL}(V)$. This contradicts the non-linearity of γ , which follows from Cryptographic Assumption (2a). Thus we will assume $k > 4$ in the rest of this section.

Note that $G = \langle T, \rho \rangle$, and $T \leq \text{Soc}(G)$, so that $G/\text{Soc}(G)$ is cyclic, spanned by ρ . Since P permutes transitively the S_i , it follows that ρ permutes *cyclically* the S_i by conjugation, that is, we may rename indices so that $S_i^\rho = \rho^{-1} S_i \rho = S_{i+1}$ for each i (and indices are taken modulo r).

Since each T_i is a group of translations, $W_i = 0T_i \subseteq 0S_i$ is a subspace of V , of order k . Since $0S_i$ has also order k , $0T_i = 0S_i$. Clearly each element of $v \in V$ can be written uniquely in the form $v = 0t$, for $t \in T$. Thus

$$v = 0t_1 t_2 \dots t_r = 0t_1 + 0t_2 + \dots + 0t_r$$

for unique $t_i \in T_i$, and

$$V = W_1 \oplus W_2 \oplus \dots \oplus W_r.$$

For each i we have also $W_i \rho = 0S_i \rho = 0S_{i+1}^{\rho^{-1}} \rho = 0\rho S_{i+1} = 0S_{i+1} = W_{i+1}$, as $0\rho = 0$. Thus ρ permutes cyclically the W_i . Now let $v \in V$, and write it as $v = w_1 + \dots + w_r$ where $w_i \in W_i$. Let $t_i \in W_i$ be such that $w_i = 0t_i$. Since the t_i are translations, we have $v = 0t_1 + 0t_2 + \dots + 0t_r = 0t_1 t_2 \dots t_r$. We have $v\rho = 0t_1 t_2 \dots t_r \rho = 0t_1^\rho t_2^\rho \dots t_r^\rho$, as $0\rho^{-1} = 0$. Since $t_i^\rho \in S_i^\rho = S_{i+1}$, there are $t'_i \in T_i$ such that $0t_i^\rho = 0t_i \rho = 0t'_{i+1} \in W_{i+1}$, and because S_i and S_j commute elementwise, we have

$$\begin{aligned} v\rho &= 0t_1^\rho t_2^\rho \dots t_r^\rho = 0t'_2 t_2^\rho \dots t_r^\rho = 0t'_2 t'_2 \dots t_r^\rho \\ &= 0t'_3 t'_2 \dots t_r^\rho = 0t'_2 t'_3 \dots t_r^\rho = \dots \\ &= 0t'_2 t'_3 \dots t'_1 = 0t'_1 + 0t'_2 + \dots + 0t'_r \\ &= 0t_r \rho + 0t_1 \rho + \dots + 0t_{r-1} \rho \\ &= w_1 \rho + w_2 \rho + \dots + w_r \rho. \end{aligned}$$

Now fix an index i , and take $u \in W_i$. We have from the above

$$v\rho = (w_1 + w_2 + \dots + w_r)\rho = w_1 \rho + w_2 \rho + \dots + w_r \rho,$$

where $w_i \rho \in W_{i+1}$, and also

$$(v + u)\rho = w_1 \rho + (w_i + u)\rho + \dots + w_r \rho$$

with $(w_i + u)\rho \in W_{i+1}$. It follows

$$(6.1) \quad (v + u)\rho + v\rho = w_i \rho + (w_i + u)\rho \in W_{i+1}.$$

Now $\rho = \gamma\lambda$, where λ is linear. Applying λ^{-1} to both sides of (6.1) we get $(v + u)\gamma + v\gamma \in W_{i+1}\lambda^{-1}$. In other words, there are subspaces $W_i, W_{i+1}\lambda^{-1}$ of V of the same dimension such that when the input difference to γ is in the first one, then the output difference is in second one. By the arguments of Section 3 (with $U = W_i$ and $U' = W_{i+1}\lambda^{-1}$), it follows that W_i is the direct sum of some of the V_j , for each i . Thus $W_2 = W_1\rho = W_1\lambda$ and $W_3 = W_2\lambda$, contradicting Cryptographic Assumption (3).

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